The Highly Collapsed Configurations of a Stellar Mass.

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(*Communicated by Professor E. A. Milne.*)

§ 1. Professor Milne in his recent paper * on "The Analysis of Stellar Structure" has put forward some essentially new considerations on the possible steady-state configurations of stellar aggregates of varying mass, luminosity, and opacity. One of the main consequences of the analysis is the explanation not only of the existence of white dwarfs—his collapsed configurations—but also of the principal physical characteristics of these configurations. The following is devoted to the development of Milne's theory of these collapsed configurations a stage further.

§ 2. Milne's estimates for the central density and temperature of these collapsed configurations indicate that in some cases we pass beyond the range of validity of the degenerate form of the Fermi-Dirac equation of state ($p = \mathrm{K}\rho^{\frac{5}{3}}$). It can be shown that the pressure of an electron gas which is highly degenerate and which has a very highly predominant relativistic-mass variation effect, takes the limiting form †

$$p = \frac{n^{\frac{4}{3}}hc}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad (1)$$

(c = velocity of light, h = Planck's constant) if the following two conditions are satisfied:—

$$n > \left(\frac{k\mathrm{T}}{hc}\right)^{3}\frac{4\pi\mathrm{G}}{3} \equiv 2 \cdot 86\mathrm{T}^{3} \qquad (\mathrm{G} = 2) \quad . \quad (\mathrm{i})$$

$$n > \frac{8\pi m^{3}c^{3}}{3h^{3}} \equiv 5 \cdot 88 \times 10^{29} \quad . \quad . \quad . \quad (\mathrm{ii}) \ddagger$$

(i) replaces Sommerfeld's degeneracy criterion, and (ii) ensures the predominance of the relativistic effect. As the condition that configurations are highly collapsed is precisely equivalent to the condition that in the central regions the electron assembly is degenerate,§ it would be sufficient to consider whether (ii) is satisfied in order to count the relativistic effect as highly predominant. Further, as L for these configurations is very small, T_c is normally of the order of 10^9,‖ and (ii) then would automatically provide for (i). (See equation (19′)).

* *M.N.*, **91**, 4–55, 1930, referred to hereafter as *loc. cit.* For a general exposition of his main ideas see *Nature*, 1931 January 3, "Stellar Structure and the Origin of Stellar Energy."

† The corresponding expression for the energy E was obtained by E. C. Stoner, *Phil. Mag.*, **9**, 944, 1930. That $p = \frac{1}{3}\mathrm{E}$ is generally true when the relativistic effect is highly predominant can easily be proved.

‡ (ii) is given in Stoner's paper; (i) now replaces Sommerfeld's criterion. These and other inequalities are briefly discussed in my paper on "The Dissociation Formula according to the Relativistic Statistics," *M.N.* (in course of publication).

§ Milne, *loc. cit.*, § 22. ‖ Milne, *loc. cit.*, p. 39.

Now the central density of a highly collapsed configuration considered as an Emden polytrope " $n = \frac{3}{2}$ " is given by *

$$\rho_c = \frac{32\pi G^3 M^2 \beta^3}{125 K_1^3 \times 7 \cdot 385} \quad . \quad . \quad . \quad . \quad (2)$$

where K_1 is the degenerate-gas constant and

$$\beta = 1 - \frac{\kappa L}{4\pi c G M} \quad . \quad . \quad . \quad . \quad (3)$$

It is clear then that the relativistic effect will be predominant in the central regions of those collapsed configurations whose masses satisfy the inequality

$$\frac{32\pi G^3 M^2 \beta^3}{125 K_1^3 \times 7 \cdot 385 \times \mu} > \frac{8\pi m^3 c^3}{3h^3}$$

or

$$M\beta^{\frac{3}{2}} > 5\left[\frac{5\mu \times 7 \cdot 385}{12}\right]^{\frac{1}{2}}\left(\frac{mcK_1}{Gh}\right)^{\frac{3}{2}} = 0 \cdot 434 \odot \quad (\text{if } \mu = 2 \cdot 5 m_H) \quad (4)$$

where μ is the mean molecular weight and $\odot$ denotes the mass of the Sun ($= 1 \cdot 985 \times 10^{33}$ gms.).

The purpose of this paper is to find out the consequences of introducing the equation of state $p = K_2 \rho^{\frac{4}{3}}$. It will be shown that we can enumerate the complete linear sequence of steady-state configurations for an *assigned* small luminosity as the mass varies, the opacity and source-strength being constant and uniform (standard model).

§ 3. *The Equations of the Problem.*—We base our subsequent discussion exclusively on the standard model as it is considerably easier to work with.

We have the following set of equations for the standard model independent of the equation of state we may adopt. (The notation is identical with that used in Milne's paper.)

$$\frac{dp}{dr} + \frac{dp'}{dr} = -\frac{GM(r)}{r^2}\rho \quad . \quad . \quad . \quad (5)$$

$$\frac{dp'}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho \quad . \quad . \quad . \quad (6)$$

$$\frac{L(r)}{M(r)} = \epsilon = \frac{L}{M}.$$

We have also the following relation between the gas kinetic-pressure (p) and the radiation-pressure p':

$$p = p'\frac{\beta}{1-\beta} \quad . \quad . \quad . \quad . \quad (7)$$

where β is defined by (3).

Case I. (*The Relativistic-degenerate Case*).—We have for the gas

* Equation (55), *loc. cit.*

kinetic-pressure, taking into account only the electronic contribution, which is certainly by far the most important

$$p = \frac{n^{\frac{4}{3}}hc}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad (1')$$

If we assume the molecular weight $\mu = 2 \cdot 5 m_H$, then

$$p = K_2 \rho^{\frac{4}{3}} . \quad . \quad . \quad . \quad . \quad . \quad (1'')$$

where

$$K_2 = \frac{hc}{8(2 \cdot 5 m_H)^{\frac{4}{3}}}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} = 3 \cdot 619 \times 10^{14} \quad . \quad . \quad (1''')$$

With the equation of state given by (1''), the equation of mechanical equilibrium reduces to

$$\frac{4K_2}{3G\beta} r^2 \rho^{-\frac{2}{3}} \frac{d\rho}{dr} = - M(r) \quad . \quad . \quad . \quad (8)$$

Remembering that

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho,$$

we have on differentiating (8)

$$\frac{4K_2}{3G\beta} \frac{d}{dr}\left(r^2 \rho^{-\frac{2}{3}} \frac{d\rho}{dr}\right) = - 4\pi r^2 \rho \quad . \quad . \quad . \quad (9)$$

Putting

$$\rho = \lambda_3 \chi^3 . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(9) reduces to

$$\frac{K_2 \lambda_3^{-\frac{2}{3}}}{\pi G\beta} \cdot \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\chi}{dr}\right) = - \chi^3 \quad . \quad . \quad . \quad (11)$$

Changing r to the variable ζ given by

$$r = \zeta\left[\frac{K_2 \lambda_3^{-\frac{2}{3}}}{\pi G\beta}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (12)$$

we have finally

$$\frac{1}{\zeta^2} \frac{d}{d\zeta}\left(\zeta^2 \frac{d\chi}{d\zeta}\right) = - \chi^3 . \quad . \quad . \quad . \quad (13)$$

which is Emden's polytropic equation with index 3.

From (8), using the values of ρ and r given by (10) and (12), we have

$$M(r) = - \frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}} \zeta^2 \frac{d\chi}{d\zeta} \quad . \quad . \quad . \quad (14)$$

It may be noted that, unlike the degenerate non-relativistic case, λ_3 has disappeared from (14).

Case II. (*The Non-relativistic-degenerate Case*).—We can use the equations given by Milne, *loc. cit.*, equations (49), (48), (50), and (47) respectively :

$$\frac{1}{\eta^2} \frac{d}{d\eta}\left(\eta^2 \frac{d\psi}{d\eta}\right) = - \psi^{\frac{3}{2}} \quad . \quad . \quad . \quad (15)$$

where
$$\rho = \lambda_2 \psi^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$r = \eta\left[\frac{5K_1}{8\pi G\beta}\lambda_2^{-\frac{1}{3}}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (17)$$

We have also
$$M(r) = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K_1}{G\beta}\right)^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}\eta^2\frac{d\psi}{d\eta} \quad . \quad . \quad . \quad (18)$$

K_1 in (17) and (18) is the degenerate-gas constant.*

§ 4. *The Equations of Fit.*—Let $r = r'$ be the radius of the surface of demarcation, *i.e.* outside $r = r'$ the distribution of density is given by a solution of Emden's equation "$n = \frac{3}{2}$," and inside $r = r'$ by a solution of Emden's equation "$n = 3$."

Outside $r = r'$ the equation of state is $p = K_1\rho^{\frac{5}{3}}$ and inside $r = r'$ we have $p = K_2\rho^{\frac{4}{3}}$. Let ρ' and T′ be the values of ρ and T at the interface, so that we have

$$K_1\rho'^{\frac{5}{3}} = K_2\rho'^{\frac{4}{3}} = \tfrac{1}{3}aT'^4\frac{\beta}{1-\beta}$$

or
$$\rho' = \left(\frac{K_2}{K_1}\right)^3; \qquad T' = \frac{K_2}{K_1}\left[\frac{K_2}{\frac{1}{3}a}\,\frac{1-\beta}{\beta}\right]^{\frac{1}{4}} \quad . \quad . \quad (19)$$

or, using numerical values,†

or
$$\left.\begin{aligned}\rho' &= 4\cdot 84 \times 10^6\\ n' &= 1\cdot 165 \times 10^{30}\end{aligned}\right\}; \qquad T' = 3\cdot 29 \times 10^9\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{4}} \quad . \quad (19')$$

Let $\phi(\eta)$ be any solution of

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -\psi^{\frac{3}{2}} \quad . \quad . \quad . \quad (20)$$

and suppose that it vanishes at $\eta = \eta_0$. Then $B^4\phi(B\eta)$ is also a solution of (20) and it vanishes at $\eta = \eta_1$, where $B\eta_1 = \eta_0$. Write $B\eta' = b_1$. By (19) and (16) we have

$$\rho' = \left(\frac{K_2}{K_1}\right)^3 = \lambda_2\psi'^{\frac{3}{2}} = \lambda_2 B^6[\phi(b_1)]^{\frac{3}{2}} \quad . \quad . \quad (21)$$

and
$$M(r') = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K_1}{G\beta}\right)^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}B^3 b_1^2\phi'(b_1) \quad . \quad . \quad (22)$$

or, eliminating $\lambda_2^{\frac{1}{2}}B^3$ by (21), we have

$$M(r') = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K_2}{G\beta}\right)^{\frac{3}{2}}\frac{b_1^2\phi'(b_1)}{[\phi(b_1)]^{\frac{3}{4}}} \quad . \quad . \quad (23)$$

Also by (17)
$$r' = \frac{b_1}{B}\left[\frac{5K_1}{8\pi G\beta}\right]^{\frac{1}{2}}\lambda_2^{-\frac{1}{6}} \quad . \quad . \quad . \quad (24)$$

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* $K_1 = 2\cdot 138 \times 10^{12}$ or $3\cdot 17 \times 10^{12}$, according as $\mu = 2\cdot 5m_H$ or $2m_H$. The former value will be used in the numerical work of this paper.

† Remembering that for collapsed configurations $\beta \sim 1$ we see that $n' = 1\cdot 165 \times 10^{30}$ provides for the condition (i) (§ 2) also.

or, eliminating $B\lambda_2^{\frac{1}{3}}$ by (21), we have

$$r' = \left[\frac{5}{8\pi G\beta K_2}\right]^{\frac{1}{2}} K_1 b_1 [\phi(b_1)]^{\frac{1}{4}} \quad . \quad . \quad . \quad (25)$$

Now let $g(\zeta)$ be any solution of the equation

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\chi}{d\zeta}\right) = -\chi^3 \quad . \quad . \quad . \quad (26)$$

which has the first zero at $\zeta = \zeta_0$. Then $Ag(A\zeta)$ is also a solution of (26) which vanishes at $\zeta = \zeta_1$, where $A\zeta_1 = \zeta_0$. Write $c_1 = A\zeta'$. We have

$$\rho' = \left(\frac{K_2}{K_1}\right)^3 = \lambda_3\chi'^3 = \lambda_3 A^3 g(c_1)^3 \quad . \quad . \quad (27)$$

By (12)

$$r' = \frac{c_1}{A}\left(\frac{K_2}{\pi G\beta}\right)^{\frac{1}{2}}\lambda_3^{-\frac{1}{3}}$$

or, eliminating $A\lambda_3^{\frac{1}{3}}$ by (27), we have

$$r' = \left[\frac{1}{\pi G\beta K_2}\right]^{\frac{1}{2}} K_1 c_1 g(c_1) \quad . \quad . \quad . \quad (28)$$

By (14) we have also

$$M(r') = -\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}} c_1^2 g'(c_1) \quad . \quad . \quad . \quad (29)$$

Now our conditions of the fitting of our two configurations are that r' given by (25) and (28) as also $M(r')$ given by (23) and (29) are identical. Equating the respective sides we find we are simply left with

$$\left(\frac{8}{5}\right)^{\frac{1}{2}} c_1 g(c_1) = b_1[\phi(b_1)]^{\frac{1}{4}} \quad . \quad . \quad . \quad (30)$$

$$\left(\frac{8}{5}\right)^{\frac{3}{2}} c_1^2 g'(c_1) = \frac{b_1^2\phi'(b_1)}{[\phi(b_1)]^{\frac{3}{4}}} \quad . \quad . \quad . \quad (31)$$

which are just Milne's equations of fit (100) and (101),* for the transition from a gaseous (Maxwellian) envelope to a degenerate core. We see therefore that when the conditions do not become so drastic as to necessitate the introduction of a homogeneous core, the analytical and the computational difficulties are reduced, as we have to solve the same set of equations for the two transitions—namely, that from a gaseous to a degenerate atmosphere, and then that from the degenerate to the relativistically degenerate atmosphere.

§ 5. *A Completely-relativistically Degenerate Configuration.*—We consider now a configuration built *entirely* on the relativistic-degenerate equation of state $p = K_2\rho^{\frac{4}{3}}$. This is therefore an Emden polytrope

* Professor Milne has since drawn my attention to the fact that this is just what we ought to expect, and that the $\left(\frac{8}{5}\right)$ occurring in (30) and (31) is just $\frac{n_1+1}{n_2+1}$, where n_1 and n_2 are the indices of the two Emden equations.

" $n = 3$ " and is similar to the Emden-Eddington diffuse configurations. But since we assume the validity of the equation of state $p = K_2\rho^{\frac{4}{3}}$ right from the boundary it is clear that if this configuration is to approximate to anything practically realisable, the central density must be sufficiently high to make the correction due to the degenerate fringe negligible. We show later that this Emden polytrope has a $\rho_c = \rho_{max}$ (the maximum density matter is capable of), in which case the correction due to the degenerate "fringe" does become negligible.

We choose λ_3 such that the value of ζ at which χ vanishes is unity, *i.e.* by (12) we choose λ_3 such that

$$r_1^2 = \frac{K_2\lambda_3^{-\frac{2}{3}}}{\pi G\beta} \quad . \quad . \quad . \quad . \quad (32)$$

where r_1 is the radius of the star. Hence

$$\lambda_3 = \left(\frac{K_2}{\pi G\beta}\right)^{\frac{3}{2}} \cdot \frac{1}{r_1^3} \quad . \quad . \quad . \quad . \quad (33)$$

The central density is given by

$$\rho_c = \lambda_3(\chi)_0^3 = \lambda_3\zeta_0^3 \quad . \quad . \quad . \quad (34')$$

or by (33)

$$\rho_c = \left(\frac{K_2}{\pi G\beta}\right)^{\frac{3}{2}}\frac{\zeta_0^3}{r_1^3} \quad . \quad . \quad . \quad . \quad (34'')$$

The central temperature would be given by

$$\tfrac{1}{3}aT_c^4\frac{\beta}{1-\beta} = K_2\rho_c^{\frac{4}{3}}$$

or

$$T_c = \left(\frac{K_2}{\frac{1}{3}a}\right)^{\frac{1}{4}}\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{4}}\left(\frac{K_2}{\pi G\beta}\right)^{\frac{1}{2}} \cdot \frac{\zeta_0}{r_1} \quad . \quad . \quad . \quad (35)$$

As is well known, for the Emden polytrope " $n = 3$ " the central density ρ_c and the mean density ρ_m are related by

$$\frac{\rho_c}{\rho_m} = -\frac{\zeta_0}{3\chi'(\zeta_0)} = 54{\cdot}36 \quad . \quad . \quad . \quad (35')$$

Finally we have a relation connecting the mass and luminosity, which is merely the condition that the whole mass shall be representable as a relativistic configuration of Emden type

$$M = -\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}}\zeta_0^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta_0} \quad . \quad . \quad . \quad (36)$$

where, since for Emden's solution $\zeta_0^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta_0} = -2{\cdot}015$, we have, introducing numerical values in (36),

$$M = 0{\cdot}9177\,\odot\beta^{-\frac{3}{2}} \equiv M_3 \quad . \quad . \quad . \quad (36')$$

we have also the limiting relation

$$\underset{L\to 0}{M} \longrightarrow \frac{2{\cdot}015 \times 4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G}\right)^{\frac{3}{2}} = 0{\cdot}92\,\odot \quad . \quad . \quad (36'')$$

§ 6. If the white dwarf under consideration could legitimately be considered as obeying down to its central regions the Emden equation "$n = \frac{3}{2}$," it is clear that ρ_c so calculated should not exceed $\rho'\left[= \left(\frac{K_2}{K_1}\right)^3\right]$. The central density of a highly collapsed configuration which is a complete Emden polytrope "$n = \frac{3}{2}$" is

$$\rho_c = \frac{32\pi G^3 M^2 \beta^3}{125 K_1^3 \times 7{\cdot}385} \quad . \quad . \quad . \quad (2)$$

We must have therefore

$$\frac{32\pi G^3 M^2 \beta^3}{125 K_1^3 \times 7{\cdot}385} \leqslant \left(\frac{K_2}{K_1}\right)^3.$$

Hence for considerations based on Emden's "$n = \frac{3}{2}$" alone to be valid we must have the inequality

$$M \leqslant 1{\cdot}214 \times 10^{33} \beta^{-\frac{3}{2}} \text{ grams} = 0{\cdot}6115\,\odot\,\beta^{-\frac{3}{2}} \;(= M_{\frac{3}{2}}, \text{ say})$$

satisfied. Hence collapsed configurations of mass *less than* $M_{\frac{3}{2}}$ are Emden polytropes "$n = \frac{3}{2}$." For $M = M_{\frac{3}{2}}$, ρ_c is just equal to our "interfacial density." If M becomes greater than $M_{\frac{3}{2}}$ the relativistic core spreads and the configurations become composite. We proceed now to the study of these composite-configurations.

§ 7. *The Composite Series I.*—In working out the composite series we again consider only Emden's solution for the relativistic core, *i.e.* we exclude for the present the possibility of the conditions becoming so drastic as to necessitate the changing over from $p = K_2\rho^{\frac{4}{3}}$ equation of state. (It will be seen that when ρ_c becomes equal to ρ_{max}—the maximum density matter is capable of—the degenerate fringe becomes negligible, and so we are not required to introduce the relativistic and the homogeneous core simultaneously.)

We have, since $\chi(0) = 1$, if χ is Emden's solution for "$n = 3$,"

$$\rho_c = \lambda_3 A^3 \quad . \quad . \quad . \quad . \quad . \quad (37)$$

or by (27)

$$\rho_c = \left(\frac{K_2}{K_1}\right)^3 \frac{1}{[g(c_1)]^3} \quad . \quad . \quad . \quad . \quad (38)$$

For the central temperature we have, since

$$\tfrac{1}{3}aT_c^4 = K_2\rho_c^{\frac{4}{3}}\frac{1-\beta}{\beta},$$

$$T_c = \left(\frac{3K_2}{a}\right)^{\frac{1}{4}}\frac{K_2}{K_1}\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{4}} \cdot \frac{1}{g(c_1)} \quad . \quad . \quad . \quad (39)$$

We have also

$$\frac{\rho'}{\rho_c} = [g(c_1)]^3\,; \qquad \frac{T'}{T_c} = g(c_1) \quad . \quad . \quad . \quad (40)$$

The radius r_1 of the whole configuration is given by

$$\frac{r'}{r_1} = \frac{\eta'}{\eta_1} = \frac{B\eta'}{B\eta_1} = \frac{b_1}{\eta_0} \quad . \quad . \quad . \quad (41)$$

or by (25)

$$r_1 = \left[\frac{5}{8\pi G\beta K_2}\right]^{\frac{1}{2}} K_1\eta_0[\phi(b_1)]^{\frac{1}{4}} \quad . \quad . \quad . \quad (42)$$

The effective temperature and the mean density are easily found to be

$$T_e = \left(\frac{L}{ac}\right)^{\frac{1}{4}}\left(\frac{8G\beta K_2}{5}\right)^{\frac{1}{4}} \cdot \frac{1}{K_1^{\frac{1}{2}}} \cdot \frac{1}{\eta_0^{\frac{1}{2}}[\phi(b_1)]^{\frac{1}{8}}} \quad . \quad . \quad (43)$$

$$\rho_m = \frac{3}{4}\left[\frac{8\pi G\beta K_2}{5}\right]^{\frac{3}{2}} \frac{M}{\pi K_1^3} \frac{1}{\eta_0^3[\phi(b_1)]^{\frac{3}{4}}} \quad . \quad . \quad . \quad (44)$$

It is not difficult to put the above equations in a form which makes it clear that as $b_1 \rightarrow \eta_0$ and $c_1 \rightarrow \zeta_0$ these composite configurations *continuously* pass over into the complete relativistic Emden polytrope " $n = 3$ " with $M = \cdot 92 \odot \beta^{-\frac{3}{2}}$. In making the reduction we make free use of (30), (31), and (42):

$$\rho_c = \left(\frac{K_2}{\pi G\beta}\right)^{\frac{3}{2}} \frac{c_1^3}{r_1^3} \cdot \frac{\eta_0^3}{b_1^3} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\frac{\rho_c}{\rho_m} = -\frac{1}{3}\frac{c_1}{g'(c_1)}\frac{M(r')}{M(r_1)} \cdot \frac{\eta_0^3}{b_1^3} \quad . \quad . \quad . \quad . \quad (45')$$

$$M(r') = -\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}} \zeta_0^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta_0}\left[\frac{c_1^2 g'(c_1)}{\zeta_0^2 g'(\zeta_0)}\right] \quad . \quad . \quad (46)$$

$$T_c = \left(\frac{K_2}{\frac{1}{3}a}\right)^{\frac{1}{4}}\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{4}}\left(\frac{K_2}{\pi G\beta}\right)^{\frac{1}{2}} \frac{c_1}{r_1}\frac{\eta_0}{b_1} \quad . \quad . \quad (46')$$

But when $b_1 \rightarrow \eta_0$ and $c_1 \rightarrow \zeta_0$ it is clear from (42) and (38) that simultaneously $r_1 \rightarrow 0$ and $\rho_c \rightarrow \infty$.* Thus the completely relativistic model considered as the limit of the composite series is a point-mass with $\rho_c = \infty$! The theory gives this result because $p = K_2\rho^{\frac{4}{3}}$ allows any density provided the pressure be sufficiently high. We are bound to assume therefore that a stage must come beyond which the equation of state $p = K_2\rho^{\frac{4}{3}}$ is not valid, for otherwise we are led to the physically inconceivable result that for $M = 0{\cdot}92 \odot \beta^{-\frac{3}{2}}$, $r_1 = 0$, and $\rho = \infty$. As we do not know physically what the next equation of state is that we are to take, we assume for definiteness the equation for the homogeneous incompressible material $\rho = \rho_{\max}$, where $\rho_{\max}$ is the maximum density of which matter is capable. The preceding analysis would then break down when ρ_c given by (38) exceeds $\rho_{\max}$. Now $\rho_{\max}$ must at the lowest estimate be of the order of 10^{12} grams cm.$^{-3}$, for if the "maximum density of matter is limited only by the sizes of the electrons and nuclei, densities of the order 10^{14} should not be impossible." † Our interfacial

* This suggests that when g is Emden's solution, φ is of the centrally condensed type.

† R. H. Fowler, *M.N.*, **87**, 114, 1926, "Dense Matter."

density is only $4 \cdot 84 \times 10^6$ and the ratio of this to the central density $\rho_c = \rho_{\max}$ is of the order of 10^{-6}, and a reference to the Emden tables shows that if for the moment we assume the star as *completely* relativistically degenerate, before we have proceeded $\frac{1}{1000}$th of the radius of the star into the interior, our assumption becomes valid. Hence the correction due to the degenerate "fringe" is negligible. We can therefore to a high degree of approximation consider as the limit of these composite series, the Emden polytrope "$n = 3$" with $\rho_c = \rho_{\max}$ and $M = 0 \cdot 92 \odot \beta^{-\frac{3}{2}}$, it being understood that $\rho_{\max}$ is sufficiently high to make the correction due to the degenerate fringe negligible. (Hence for highly collapsed configurations of mass greater than $0 \cdot 92 \odot \beta^{-\frac{3}{2}}$ we can neglect the degenerate fringe.) Thus the highly collapsed configurations for which $0 \cdot 612 \odot \beta^{-\frac{3}{2}} < M < 0 \cdot 92 \odot \beta'^{-\frac{3}{2}}$ are composite, consisting of a degenerate envelope surrounding a relativistic core. These composite configurations are bordered on either side by two completely determinable configurations—on the one side by an Emden polytrope "$n = \frac{3}{2}$," and on the other by an Emden polytrope "$n = 3$" with $\rho_c = \rho_{\max}$.

§ 8. *Composite Series II.*—So far we have considered only the Emden solutions for the relativistic core, and we have been able to specify the steady-state configurations only for $M \leqslant M_3$. Further, $M = M_3$ corresponds to a high degree of approximation to an Emden-polytrope "$n = 3$" with $\rho_c = \rho_{\max}$. Hence for $M > M_3$ we should expect the homogeneous core to have spread out. We should therefore consider the fit of a relativistic envelope surrounding a homogeneous core with $\rho = \rho_{\max}$, the core and the envelope being continuous at the interface. If $r = r''$ (where $\zeta = \zeta''$ and $\chi = \chi''$) is the radius of the surface of demarcation, the equation of fit is found to reduce to

$$\tfrac{1}{3}\zeta''^3\chi''^3 = -\zeta''^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta''} \quad . \quad . \quad . \quad (47)$$

It can easily be shown that if χ be an Emden or a centrally condensed type of solution, then (47) has no roots, except that in the former case we have the trivial solution $\zeta'' = 0$. Hence the introduction of the homogeneous core compels us to a consideration of only the collapsed-type solutions for χ.

Now by (36)

$$-\zeta_0^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta_0} = \frac{M}{\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}}} = \frac{1}{C_2^{\frac{1}{2}}} \quad . \quad . \quad (48)$$

where C_2 can now be called the "discriminant" of the relativistic standard model for $M > M_3$. It may be noted here that C_2 is primarily a function *only* of M, since the hypothesis that the configurations are highly collapsed provides us with $\beta \sim 1$ in any case. We can write (48) differently as

$$-\zeta_0^2\left(\frac{d\chi}{d\zeta}\right)_{\zeta=\zeta_0} = \frac{M}{M_3} \times 2 \cdot 015 \quad . \quad . \quad . \quad (49)$$

where 2·015 is the corresponding boundary value for the Emden solution. Hence for $M > M_3$ we clearly see from (49) that the solutions for χ now belong to the collapsed family. Thus the condition that $M > M_3$ is precisely equivalent to the condition that there is a homogeneous core. By methods similar to § 4 we easily obtain

$$M(r'') = \tfrac{4}{3}\pi r''^3 \rho_{\max} = -\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{K_2}{G\beta}\right)^{\frac{3}{2}} c_1^2 g'(c_1) \quad . \quad . \quad (50)$$

$$r_1 = \frac{r''\zeta_0}{c_1} \quad . \quad . \quad . \quad . \quad (51)$$

where $c_1 = A\zeta''$ and r_1 is the radius of the whole configuration. Comparing (50) with (49) we see that as M increases beyond M_3 the collapse proceeds further and further till finally, when $M \to \infty$, $c_1 \to \zeta_0$, and $r_1 \to r''$, and the whole configuration has completely "collapsed" into one mass of incompressible matter at the highest density matter is capable of. We have in the limit, so to say, a "*solid star*."

Further, if $g(c_1)$ corresponds to the Emden solution there is only one "trivial" root for (47), namely, $c_1 = 0$, *i.e.* the central density of this completely relativistic Emden polytrope is just equal to $\rho_{\max}$ and the radius of the star is then obviously given by

$$\frac{\frac{4}{3}\pi r_1^3 \rho_{\max}}{54\cdot 36} = 0\cdot 92 \odot \beta^{-\frac{3}{2}} \quad . \quad . \quad . \quad (52)$$

Thus this Composite Series II. joins continuously the Composite Series I. (§ 7) and the Emden polytrope "$n = 3$" with $\rho_c = \rho_{\max}$, and $M = \cdot 92 \odot \beta^{-\frac{3}{2}}$ is the *common limit* of both the series. We have therefore the following complete classification of the highly collapsed configurations ($L << L_0$, $\beta \sim 1$) for M *considered as a variable taking the whole range of values.*

Mass.	Description.
Class I.—$M < 0\cdot 61 \odot \beta^{-\frac{3}{2}}$	Emden polytropes "$n = \frac{3}{2}$."
$M_{\frac{3}{2}} = M = \cdot 61 \odot \beta^{-\frac{3}{2}}$	An Emden polytrope $n = \frac{3}{2}$ with $\rho_c = \left(\frac{K_2}{K_1}\right)^3$.
Class II.—$0\cdot 61 \odot \beta^{-\frac{3}{2}} < M < 0\cdot 92 \odot \beta'^{-\frac{3}{2}}$	Composite I.—Degenerate envelope surrounding a homogeneous core.
$M_3 = M = 0\cdot 92 \odot \beta'^{-\frac{3}{2}}$	Approximately an Emden polytrope "$n = 3$" with $\rho_c = \rho_{\max}$.
Class III.—$M > \cdot 92 \odot \beta'^{-\frac{3}{2}}$	Composite II.—relativistic envelope and homogeneous core.
$M \to \infty$	Completely homogeneous ($\rho = \rho_{\max}$).

That we are thus able to enumerate definitely the steady-state configurations for the whole range of M appears to be in complete conformity with the general scheme of Milne's ideas.

To apply the above classification to the known white dwarfs—o_2 Eridani B, Procyon B, and Van Maanen's star possibly belong to Class I. That the companion of Sirius is in Class II. is also likely.

But it appears that no white dwarf has yet been discovered which has a homogeneous core at the centre. This classification is made with caution, since, though they are certainly of the collapsed type, they are by no means "*highly*" collapsed, for then L and T_e would be so small that we could not see the stars.

§ 9. *Summary.*—In this paper Milne's theory of collapsed configurations is developed a stage further, the essential refinement being in the introduction of a relativistically degenerate core with the equation of state $p = K_2\rho^{\frac{4}{3}}$. This enables an enumeration to be made of the steady-state configurations for the whole range of M considered as a variable with $L \sim 0$. The classification arrived at is shown in the table in § 8.

In conclusion, I wish to record my best thanks to Professor Milne for much valuable advice and criticism during the course of the work.

The Possible Solutions of the "Equations of Fit" on the Standard Model. By B. Strömgren, Ph.D.

(*Communicated by E. A. Milne.*)

1. In a recent paper * Professor E. A. Milne has reconsidered the problem of determining the structure of the stars. A full treatment is given of stars built on the standard model, κ = constant, ϵ = constant. Professor Milne's treatment of this model is, however, not yet completed, as he himself states, so far as the actual fit of "$n = 3$"-polytropic solutions (f) on to "$n = \frac{3}{2}$"-polytropic solutions (ϕ) is concerned. In fact, Professor Milne awaits the construction of a complete set of tables of the functions f and ϕ.

It is shown in the present note, however, that a discussion of the point mentioned is possible without these tables, certain general theorems on polytropic solutions established by Mr. R. H. Fowler † being sufficient.

2. The discussion is based on the equations of fit derived by Professor Milne and given in the paper quoted as equations (100) and (101). The line of thought leading to (100) and (101) may be briefly stated as follows. For a given mass M a solution is started from the boundary inwards; this solution is an "$n = 3$"-polytropic solution which is definite for definite values of the radius r_1 and the discriminant C depending on M, L, and κ (the nature of the solution depends on C, solutions with the same C and different r_1 being homologous). At a certain definite point $\rho T^{-\frac{3}{2}}$ reaches a value where degeneracy sets in. The value of the radius r_1, the remaining mass M(r), and the density ρ at this point are definite functions of M, C, and r_1. These values, which may be denoted by r', M(r'), and ρ', now determine the "$n = \frac{3}{2}$"-polytropic solution that fits on to the selected "$n = 3$"-solution.

* *M.N.*, **91**, 4, 1930. † *M.N.*, **91**, 63, 1930.